

# MODULAR DECOMPOSITION OF THE ORLIK-TERAO ALGEBRA OF A HYPERPLANE ARRANGEMENT

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**ABSTRACT.** Let  $\mathcal{A}$  be a collection of  $n$  linear hyperplanes in  $\mathbb{k}^\ell$ , where  $\mathbb{k}$  is an algebraically closed field. The Orlik-Terao algebra of  $\mathcal{A}$  is the subalgebra  $R(\mathcal{A})$  of the rational functions generated by reciprocals of linear forms vanishing on hyperplanes of  $\mathcal{A}$ . It determines an irreducible subvariety  $Y(\mathcal{A})$  of  $\mathbb{P}^{n-1}$ . We show that a flat  $X$  of  $\mathcal{A}$  is modular if and only if  $R(\mathcal{A})$  is a split extension of the Orlik-Terao algebra of the subarrangement  $\mathcal{A}_X$ . This provides another refinement of Stanley's modular factorization theorem [Sta72] and a new characterization of modularity, similar in spirit to the fibration theorem of [Par00].

We deduce that if  $\mathcal{A}$  is supersolvable, then its Orlik-Terao algebra is Koszul. In certain cases, the algebra is also a complete intersection, and we characterize when this happens.

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## 1. INTRODUCTION

We begin with a brief description of the main constructions appearing in our paper.

**1.1. Algebras of reciprocals, and reciprocal planes.** Let  $\mathbb{k}$  be an algebraically closed field, and let  $\mathcal{B}_n = \{\hat{H}_1, \dots, \hat{H}_n\}$  denote the set of coordinate hyperplanes in  $\mathbb{k}^n$ . Let  $V$  be a linear subspace of  $\mathbb{k}^n$  of dimension  $\ell$ , with the property that  $V \not\subseteq \hat{H}_i$  for any  $i$ . Consider the set of hyperplanes  $\mathcal{A} = \{H_1, \dots, H_n\}$  in  $V$ , where  $H_i = \hat{H}_i \cap V$ . The set  $\mathcal{A}$  is a central, essential hyperplane arrangement of rank  $\ell$ , and any such arrangement arises this way. Our default reference for facts about hyperplane arrangements will be the book [OT92].

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Let  $f: V \hookrightarrow \mathbb{k}^n$  denote the inclusion. Then  $f_i \in \mathbb{k}[V]$  is a linear map for which  $H_i = \ker f_i$ , for  $1 \leq i \leq n$ . Our main object of study in this paper is the following.

**Definition 1.1.** The Orlik-Terao algebra of  $\mathcal{A}$  is the subalgebra of  $\mathbb{k}(V)$  generated by reciprocals of the linear polynomials defining the hyperplanes:

$$R(\mathcal{A}) := \mathbb{k}[1/f_1, \dots, 1/f_n].$$

This algebra and certain Artinian quotients of it first appeared in work of Orlik and Terao [OT94], in the context of hypergeometric functions.

Let  $M(\mathcal{A}) = V - \bigcup_i H_i$  denote the complement of the hyperplane arrangement  $\mathcal{A}$ . Then  $M(\mathcal{A}) = V \cap (\mathbb{k}^*)^n$ , where  $(\mathbb{k}^*)^n$  is both an algebraic torus and the complement of the arrangement  $\mathcal{B}_n$ . Let  $i: \mathbb{k}^n \rightarrow \mathbb{k}^n$  denote the Cremona transformation given by  $i(y) = (y_1^{-1}, \dots, y_n^{-1})$ . Clearly the restriction of  $i$  to the torus  $(\mathbb{k}^*)^n$  is a regular map, and  $M(\mathcal{A}) \cong i(M(\mathcal{A}))$ . By construction, the Orlik-Terao algebra is the coordinate ring of the closure

$$Y(\mathcal{A}) := \overline{i(M(\mathcal{A}))}$$

in  $\mathbb{k}^n$ . We will call the variety  $Y(\mathcal{A})$  the *reciprocal plane* of  $\mathcal{A}$ , following the terminology of [SSV11], where it plays a role in understanding the entropic discriminant.

Various authors have studied the Orlik-Terao algebra and reciprocal plane: see, for example, [PS06, ST09, Loo03, HT03]. The construction has received renewed attention recently in [HK11, SSV11]. A number of basic properties are known: for example, the Cohen-Macaulay property [PS06], explicit equations for  $Y(\mathcal{A})$  (see §2.2), and the Hilbert series of  $R(\mathcal{A})$ , given in terms of matroid combinatorics:

$$(1) \quad h(R(\mathcal{A}), t) = \pi(\mathcal{A}, t/(1-t)),$$

where  $\pi(\mathcal{A}, t)$  is the Poincaré polynomial of the arrangement. This was proven by Terao [Ter02] for  $\mathbb{k}$  of characteristic zero, and by Berget [Ber10] in general.

**1.2. Modular factorizations.** Let  $L(\mathcal{A})$  denote the intersection lattice of an arrangement  $\mathcal{A}$ . A subspace  $X \in L(\mathcal{A})$  is said to be *modular* if  $X + Y \in L(\mathcal{A})$  for all subspaces  $Y \in L(\mathcal{A})$ . If  $X \in L(\mathcal{A})$  is modular, then Stanley's Factorization Theorem [Sta72] states that the Poincaré polynomial  $\pi(\mathcal{A}, t)$  is divisible by  $\pi(\mathcal{A}_X, t)$ . In fact, a result of Brylawski [Bry75] accounts for the quotient: he shows that, if  $X$  is modular, then

$$(2) \quad \pi(\mathcal{A}, t) = \pi(\mathcal{A}_X, t) \pi(\overline{T}_X(M(\mathcal{A})), t)/(1+t),$$

where  $\overline{T}_X(M(\mathcal{A}))$  denotes the complete principal truncation of the matroid  $M(\mathcal{A})$  of  $\mathcal{A}$  at  $X$ . (We refer to [Oxl11] for definitions from matroid theory.)

Falk and Proudfoot [FP02] showed that, for complex arrangements, the factorization (2) has a topological explanation. For any  $X \in L(\mathcal{A})$ , let  $p_X: V \rightarrow V/X$  denote the projection of  $V \cong \mathbb{C}^\ell$  onto its quotient by the linear space  $X$ . Then  $p_X$  restricts to a map  $p_X|_{M(\mathcal{A})}: M(\mathcal{A}) \rightarrow M(\mathcal{A}_X)$ . If, moreover,  $X$  is modular, the restriction is a locally trivial fibre bundle (shown by Terao [Ter86] in the case where  $X$  is a coatom, and Paris [Par00] in general: a detailed proof appears in [FP02].)

The fibres  $\mathcal{A}_v$  are homeomorphic to the (projective) complement of a realization of the complete principal truncation  $\overline{T}_X(M(\mathcal{A}))$ , (by [FP02, Th. 2.1]). Since the Poincaré

polynomial of a complex arrangement counts the Betti numbers of its complement, Stanley and Brylawski's factorization (2) is equivalent to the fact that the Serre spectral sequence of the fibration sequence

$$(3) \quad \mathbb{P}M(\mathcal{A}_v) \longrightarrow M(\mathcal{A}) \xrightarrow{p} M(\mathcal{A}_X)$$

degenerates at  $E_2$ .

The projection  $p$  also induces a map  $Y(\mathcal{A}) \rightarrow Y(\mathcal{A}_X)$  of reciprocal planes, and of coordinate rings  $R(\mathcal{A}_X) \rightarrow R(\mathcal{A})$ . One of our main results is that, if  $X$  is modular, then  $R(\mathcal{A})$  is a free  $R(\mathcal{A}_X)$ -module. More precisely, we give an isomorphism of  $R(\mathcal{A}_X)$ -modules,

$$(4) \quad R(\mathcal{A}) \cong R(\mathcal{A}_X) \otimes_{\mathbb{k}} R_{[n]-[X]}(\mathcal{A}),$$

where the algebra  $R_{[n]-[X]}(\mathcal{A})$  denotes the coordinate ring of the fibre over zero (Theorem 3.15). In order to interpret this algebra, we introduce a *relative* Orlik-Terao algebra  $R_H(\mathcal{A})$ , for any  $H \in \mathcal{A}$  (Definition 3.1). The fibre arrangement  $\mathcal{A}_v$  comes with a distinguished hyperplane  $X$ , and we show (Theorem 3.13) that

$$R_{[n]-[X]}(\mathcal{A}) \cong R_X(\mathcal{A}_v).$$

Then Stanley's formula (2) appears by comparing Hilbert series under the isomorphism (4), using a relative version of Terao's formula (1).

The isomorphism (4) has a combinatorial explanation. Proudfoot and Speyer [PS06] showed that the Orlik-Terao algebra of an arrangement  $\mathcal{A}$  is a flat deformation of the Stanley-Reisner ring of the broken circuit complex  $\mathbf{bc}(\mathcal{A})$  of  $\mathcal{A}$ . Brylawski and Oxley [BO81, Th. 1.6] found  $X$  is a modular flat of  $\mathcal{A}$  if and only if the broken circuit complex decomposes as a join of induced subcomplexes:

$$(5) \quad \mathbf{bc}(\mathcal{A}) \cong \mathbf{bc}(\mathcal{A})|_{[X]} * \mathbf{bc}(\mathcal{A})|_{[n]-[X]}.$$

We show (Theorem 3.9) that a flat  $X$  is modular if and only if  $\mathbf{bc}(\mathcal{A})|_{[n]-[X]} \cong \mathbf{bc}_0(\overline{T}_X(M(\mathcal{A})))$ , where  $\mathbf{bc}_0$  denotes the reduced broken circuit complex (§2.2): this is a self-contained combinatorial result that seems to have been anticipated in the introduction to [Bry77], but the statement appears to be new. Then, since a join of simplicial complexes gives rise to a tensor product of Stanley-Reisner rings, the decomposition (4) can be obtained by a deformation argument, using [PS06].

In Section 4, we extract some consequences of our modular decomposition. An arrangement  $\mathcal{A}$  is *supersolvable* if there exists a maximal modular chain in  $L(\mathcal{A})$ : that is, modular subspaces  $X_i \in L(\mathcal{A})$ , for  $1 \leq i \leq \ell$ , for which  $X_1 < \dots < X_\ell$ . We show (Theorem 2.8) that, if  $\mathcal{A}$  is supersolvable, then  $R(\mathcal{A})$  is a Koszul algebra. We also consider a further generalization to hypersolvable arrangements (introduced by Jambu and Papadima [JP98]), and compute Poincaré-Betti series for  $R(\mathcal{A})$  explicitly in the special case of generic arrangements (§4.4).

Again, we note a parallel with the topology of the complement. The cohomology ring of the complement  $M(\mathcal{A})$  (the *Orlik-Solomon algebra*) has a presentation which is similar to that of  $R(\mathcal{A})$ . By way of comparison, this algebra is a deformation of the *exterior* Stanley-Reisner ring of  $\mathbf{bc}(\mathcal{A})$ . Shelton and Yuzvinsky [SY97] showed the

cohomology ring is Koszul if  $\mathcal{A}$  is supersolvable, so our Theorem 2.8 may be regarded as a commutative analogue.

Complete intersections generated by quadrics are known to be Koszul. Typically, however,  $R(\mathcal{A})$  is not a complete intersection, even if  $\mathcal{A}$  is supersolvable. In Section 5, we characterize those arrangements for which  $R(\mathcal{A})$  is a quadratic complete intersection. Using results of Falk [Fal02] as well as Sanyal, Sturmfels, and Vinzant [SSV11], we find that such arrangements are exactly the supersolvable arrangements with exponents at most 2.

## 2. BACKGROUND

**2.1. Projection to closed subarrangements.** Here, we recall in some more detail the results of [FP02, Ter86, Par00]. As in §1.1, let  $\mathcal{A}$  denote a central, essential arrangement of  $n$  hyperplanes in an  $\ell$ -dimensional linear subspace  $V$  of  $\mathbb{k}^n$ . A hyperplane arrangement can be regarded as a linear realization of a matroid without loops, and we will denote the underlying matroid of  $\mathcal{A}$  by  $M(\mathcal{A})$ . The diagonal action of  $\mathbb{k}^*$  on  $\mathbb{k}^n$  restricts to  $M(\mathcal{A}) \subseteq V$ : we will let  $\mathbb{P}\mathcal{A}$  denote the corresponding set of projective hyperplanes in  $\mathbb{P}V$ , and let  $\mathbb{P}M(\mathcal{A})$  denote their complement.

We will order the hyperplanes of  $\mathcal{A}$  and take the underlying set of  $M(\mathcal{A})$  to be the set  $[n] := \{1, \dots, n\}$ , regarded as integers indexing ordered hyperplanes. We will abuse notation slightly and regard  $L(\mathcal{A})$  both as the intersection lattice of  $\mathcal{A}$  and the lattice of flats of  $M(\mathcal{A})$ : when the distinction is required, we write  $[X] := \{i \in [n] : H_i \leq X\}$  for  $X \in L(\mathcal{A})$ . The *rank* of a flat is its codimension in  $V$ : let  $L_p(\mathcal{A})$  denote the flats of rank  $p$ . *Coatoms* are flats of rank  $\ell - 1$ . Let  $\mathcal{A}_X$  denote the subarrangement of  $\mathcal{A}$  indexed by  $[X]$ , regarded as a hyperplane arrangement in the linear space  $V/X$ . Its intersection lattice is the lower interval  $[V, X]$  of  $L(\mathcal{A})$ . Let  $p_X : \mathbb{k}^n \rightarrow \mathbb{k}^{[X]}$  denote the coordinate projection given by deleting coordinates for hyperplanes  $H \not\leq X$ .

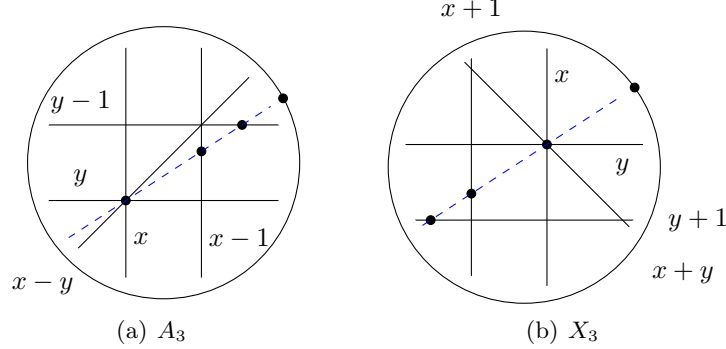
For a point  $y \in V$ , we note  $p_X(y) = 0$  if and only if  $y_i = 0$  for all hyperplanes  $H_i \leq X$ . Thus we may identify  $p_X(V)$  with  $V/X$ , and restrict  $p_X$  further to hyperplane complements,  $p_X : M(\mathcal{A}) \rightarrow M(\mathcal{A}_X)$ . The map  $p_X$  is compatible with the  $\mathbb{k}^*$  action, so it induces a map  $\bar{p}_X : \mathbb{P}M(\mathcal{A}) \rightarrow \mathbb{P}M(\mathcal{A}_X)$ . Let  $Q = \prod_{i=1}^n f_i$ . For convenience, order the hyperplanes so that  $H_i \leq X$  if and only if  $1 \leq i \leq n_X$ , where  $n_X = |\mathcal{A}_X|$ , and let  $Q_X = \prod_{i=1}^{n_X} f_i$ . Then, as schemes,  $\mathbb{P}M(\mathcal{A}) = \text{Proj}(\mathbb{k}[V]_Q)$ , and  $\mathbb{P}M(\mathcal{A}_X) = \text{Proj}(\mathbb{k}[V/X]_{Q_X})$ .

We note that, for any (reduced) point  $v \in \mathbb{P}M(\mathcal{A}_X)$ , the fibre over  $v$  is the complement of a hyperplane arrangement: consider the point projectively as a map  $v : \mathbb{P}_{\mathbb{K}}^0 \rightarrow \mathbb{P}M(\mathcal{A}_X)$ , for some extension  $\mathbb{K}$  of  $\mathbb{k}$ , given by a graded homomorphism  $v^* : \mathbb{k}[V/X]_{Q_X} \rightarrow \mathbb{K}[t]$ . Then the homogeneous coordinate ring of  $p^{-1}(v)$  is

$$\mathbb{k}[V]_Q \otimes_{\mathbb{k}[V/X]_{Q_X}} \mathbb{K}[t] \cong \mathbb{K}[\mathbb{A}^1 \times X]_{Q_v},$$

where the polynomial  $Q_v$  may be chosen to be the reduced image of  $Q$  in the tensor product.

We make the following definition, noting that our notation differs slightly from that of [FP02], in that our arrangements are always central.

FIGURE 1. Typical fibres for  $x = y = 0$ 

**Definition 2.1.** For each point  $v \in \mathbb{P}M(\mathcal{A}_X)$ , let  $\mathcal{A}_{v,X}$  denote the arrangement in  $\mathbb{A}^1 \times X$  defined by  $Q_v$ . We will simply write  $\mathcal{A}_v$  when the choice of  $X$  is understood. By construction,  $\mathbb{P}M(\mathcal{A}_{v,X}) = p_X^{-1}(v)$ . We take the underlying set of its matroid to be  $\{0\} \cup ([n] - [X])$ .

**Example 2.2.** Consider the rank-3 arrangements  $A_3$  and  $X_3$  given, respectively, by defining equations  $Q_1 = xyz(x-y)(x-z)(y-z)$  and  $Q_2 = xyz(x+y)(x+z)(y+z)$ . In each case, let  $X$  be the linear subspace given by  $x = y = 0$ , so that  $\mathcal{A}_X$  is the rank-2 arrangement of 3 lines, for  $\mathcal{A} = A_3$  and  $\mathcal{A} = X_3$ .

If  $v = [\alpha : \beta] \in \mathbb{P}M(\mathcal{A}_X)$  is a closed point and  $\mathcal{A} = A_3$ , then  $Q_v = ctz(z-\alpha t)(z-\beta t)$ , where  $c$  is a unit, and the arrangement  $\mathbb{P}\mathcal{A}_v$  consists of four points in  $\mathbb{P}_{\mathbb{K}}^1$ . On the other hand, if  $\mathcal{A} = X_3$ , then  $Q_v = ctz(z+\alpha t)(z+\beta t)$ , and  $\mathbb{P}\mathcal{A}_v$  consists of four points in  $\mathbb{P}_{\mathbb{K}}^1$  as long as  $\alpha \neq \beta$ . Figure 1 shows typical fibres for both arrangements, where each complement  $\mathbb{P}M \subseteq \mathbb{P}^2$  is drawn in the affine chart with  $z = 1$ .

If  $\eta$  is the generic point in  $\mathbb{P}M(\mathcal{A}_X)$ , let  $\mathbb{K} = \mathbb{k}(V/X)_0$  and map  $\mathbb{k}[V/X] \rightarrow \mathbb{K}[t]$  by  $x \mapsto t$ ,  $y \mapsto (y/x)t$ . Then for  $\mathcal{A} = A_3$ ,  $Q_\eta$  is a unit multiple of  $tz(z-t)(z-(y/x)t)$ , so  $\mathbb{P}\mathcal{A}_\eta$  consists of four points in  $\mathbb{P}_{\mathbb{K}}^1$ , and similarly for  $\mathcal{A} = X_3$ .  $\diamond$

In general, then, the underlying matroid of the arrangement  $\mathcal{A}_v$  depends on the choice of  $v$ . The typical value, however, is the *complete principal truncation* of  $M(\mathcal{A})$  with respect to  $X$ . We denote it by  $\overline{T}_X(M(\mathcal{A}))$ : see, e.g., [Oxl11, p. 379].

**Proposition 2.3.** *For any flat  $X$  of an arrangement  $\mathcal{A}$ , there is a dense open subscheme  $U$  of  $\mathbb{P}M(\mathcal{A}_X)$  for which  $M(\mathcal{A}_{v,X}) = \overline{T}_X(M(\mathcal{A}))$ , for all points  $v$  in  $U$ .*

*Proof.* As in [FP02, Thm. 2.1], an arrangement  $\mathcal{A}_{v,X}$  is a realization of the matroid  $\overline{T}_X(M(\mathcal{A}))$  as long as some finite list of determinants depending on  $v$  are all nonzero.  $\square$

**Definition 2.4.** In particular, if  $\eta \in \mathbb{P}M(\mathcal{A}_X)$  is the generic point, we always have  $M(\mathcal{A}_{\eta,X}) = \overline{T}_X(M(\mathcal{A}))$ . Let  $\overline{T}_X(\mathcal{A}) = \mathcal{A}_{\eta,X}$ . This is an arrangement over the field  $\mathbb{k}(V/X)_0$ , and is in some sense the canonical realization of  $\overline{T}_X(M(\mathcal{A}))$ , given  $\mathcal{A}$ .

In the complex analytic case, one can say more, provided that  $X$  is modular.

**Theorem 2.5** (Theorem 2.4, [FP02]). *If  $\mathcal{A}$  is a complex arrangement and  $X$  is modular, the restriction of  $p_X$  to  $M(\mathcal{A})$  is a fibre bundle projection. The fibres are homeomorphic to the complement of  $\mathbb{P}\mathcal{A}_{v,X}$ , for any choice of  $v \in M(\mathcal{A}_X)$ .*

In the (important) special case where  $X$  is a modular coatom,  $\mathcal{A}_{v,X}$  is an arrangement of lines in  $\mathbb{C}^2$ , so the fibre is the complement of finitely many points in  $\mathbb{P}^1$ .

**2.2. Broken circuits and the Orlik-Terao algebra.** For any arrangement  $\mathcal{A}$ , Proudfoot and Speyer [PS06] showed that the Orlik-Terao algebra (Definition 1.1) admits the following presentation, for which the reader is also referred to [ST09, Prop. 2.1]. Let  $S := \mathbb{k}[y_1, \dots, y_n]$  denote the coordinate ring of  $\mathbb{k}^n$ . Then the inclusion  $Y(\mathcal{A}) \subseteq \mathbb{k}^n$  induces a surjective homomorphism

$$(6) \quad \mathbb{k}[y_1, \dots, y_n] \rightarrow R(\mathcal{A})$$

sending  $y_i$  to  $1/f_i$ , for  $1 \leq i \leq n$ . Let  $I(\mathcal{A})$  denote the kernel of the map (6).

For any  $c \in \mathbb{k}^n$ , denote its support by  $[c] \subseteq [n]$ . The *circuits* of  $\mathcal{A}$  are those  $c \in \mathbb{k}^n$  for which  $\sum_{i=1}^n c_i f_i = 0$  and  $[c]$  is minimal. If  $c$  is a circuit, the element

$$(7) \quad r_c := \sum_{i \in [c]} c_i \prod_{j \in [c] - \{i\}} y_j$$

is easily seen to be in  $I(\mathcal{A})$ .  $c$  is determined up to a nonzero scalar multiple by its support: if  $C$  is a circuit of  $M(\mathcal{A})$ , let  $r_C = r_c$  where  $C = [c]$  and  $r_c$  is monic. Let  $\mathcal{C}(M(\mathcal{A}))$  denote the set of circuits of  $M(\mathcal{A})$ .

The relations  $\{r_c\}$  are determinantal.

**Proposition 2.6.** *Let  $X$  be a flat of  $\mathcal{A}$  of rank  $k$ , and  $n_X = |X|$ . If  $n_X > k \geq 1$ , let*

$$A_X = (\partial_j \log(f_i/f_{n_X}))_{1 \leq i \leq n_X-1, 1 \leq j \leq k}$$

where  $\{x_1, \dots, x_k\}$  generate  $\mathbb{k}[X]$ , and  $\partial_j := \partial/\partial x_j$ . Regarded as a matrix with (linear) entries in  $S$ ,

$$\text{Fitt}_k(A_X) = (r_C : C \in \mathcal{C}(M(\mathcal{A})) \text{ and } \text{cl}(C) = [X]),$$

where  $\text{cl}(C)$  denotes the smallest flat containing  $C$ .

*Proof.* By restricting to a closed subarrangement if necessary, we may assume  $X$  has rank  $k = \ell$  and  $n_X = n$ . If  $\mathcal{A}$  is not the Boolean arrangement, then  $\ell < n$ . Let  $J$  be the  $n \times \ell$  matrix whose  $(i, j)$  entry is  $\partial_j f_i$ , and let  $\mathbf{v} = (f_1, \dots, f_n)^T$ . Since  $f_i$  is linear for  $1 \leq i \leq n$ ,  $J(x_1, \dots, x_\ell)^T = \mathbf{v}$ . Let  $J'$  denote the  $n \times (\ell + 1)$  matrix obtained from  $J \mid \mathbf{v}$  by dividing row  $i$  by  $f_i$ . Evaluated on  $M(\mathcal{A})$ , the maximal minors of  $J'$  vanish. In fact, if  $C$  is a circuit of rank  $\ell$ ,  $r_C$  is a unit multiple of the minor  $J'_{C, [\ell+1]}$ . Now subtract row  $n$  of  $J'$  from each row  $i$  for  $1 \leq i \leq n - 1$ : the result may be written

$$\left( \begin{array}{ccc|c} \partial_1 \log(f_1/f_n) & \cdots & \partial_\ell \log(f_1/f_n) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \partial_1 \log(f_{n-1}/f_n) & \cdots & \partial_\ell \log(f_{n-1}/f_n) & 0 \\ \hline \partial_1 \log f_n & \cdots & \partial_\ell \log f_n & 1 \end{array} \right)$$

and the claim follows.  $\square$

These are, in fact, the only relations:

**Theorem 2.7** (Theorem 4, [PS06]). *If  $\mathcal{A}$  is an arrangement of  $n$  hyperplanes, we have  $R(\mathcal{A}) \cong \mathbb{k}[y_1, \dots, y_n]/I(\mathcal{A})$ , where  $I(\mathcal{A}) = (r_c : c \text{ is a circuit of } \mathcal{A})$ . Moreover, the relations  $\{r_c\}$  form a universal Gröbner basis for  $R(\mathcal{A})$  for which*

$$(8) \quad \text{In } R(\mathcal{A}) \cong \mathbb{k}[y_1, \dots, y_n]/J_{\mathbf{bc}(\mathcal{A})},$$

where  $J_{\mathbf{bc}(\mathcal{A})}$  is the Stanley-Reisner ideal of the broken circuit complex with respect to the given (arbitrary) ordering.

To expand on the second statement, we recall the definition of the broken circuit complex. Fix any order of the hyperplanes  $\mathcal{A}$ . A subset of  $[n]$  is a *broken circuit* if it is of the form  $C - \{i\}$ , where  $C$  is a circuit and  $i$  is its least element. A subset  $I$  of  $[n]$  is called a *nbc-set* if it does not contain a broken circuit. Clearly, nbc-sets are independent. They form an abstract simplicial complex on the vertex set  $[n]$ , denoted  $\mathbf{bc}(\mathcal{A})$  and called the *broken circuit complex*. Similarly, the *reduced broken circuit complex*, denoted  $\mathbf{bc}_0(\mathcal{A})$ , consists of all subsets of  $[n] - \{e\}$  not containing a broken circuit, where  $e$  is the least element: see [Bjö92, §7.4] and [Bry77] for reference. In particular, the complexes  $\mathbf{bc}(\mathcal{A})$  and  $\mathbf{bc}_0(\mathcal{A})$  are pure of dimension  $\ell$  and  $\ell - 1$ , respectively, where  $\ell$  is the rank of  $\mathcal{A}$ .

If  $\mathcal{A}$  is supersolvable, then the broken circuit complex decomposes inductively as a join of zero-dimensional complexes, by (5), so in this case  $\mathbf{bc}(\mathcal{A})$  is a flag complex: that is, minimal non-simplices have two vertices. Consequently, the Stanley-Reisner ideal  $J_{\mathbf{bc}(\mathcal{A})}$  is generated in degree 2, which is to say that  $R(\mathcal{A})$  is a  $G$ -algebra: that is, it possesses a quadratic Gröbner basis. This has the following consequence (see [PP05]):

**Theorem 2.8.** *If an arrangement  $\mathcal{A}$  is supersolvable, then  $R(\mathcal{A})$  is a Koszul algebra.*

We will refine this result in Section 4.

### 3. AN ALGEBRA FACTORIZATION

The goal of this section is to show that the Orlik-Terao algebra  $R(\mathcal{A})$  is a split extension of  $R(\mathcal{A}_X)$  if and only if  $X$  is a modular flat of an arrangement  $\mathcal{A}$ , Theorem 3.15. The algebra decomposition comes from a reciprocal plane analogue of the Modular Fibration Theorem 2.5. It depends on the following combinatorial characterization of modularity: we show that the (reduced) broken circuit complex of the complete principal truncation is the subcomplex  $\mathbf{bc}(\mathcal{A})|_{[n]-[X]}$  exactly when  $X$  is modular (Theorem 3.9).

**3.1. Coordinate projections and intersections.** To resume the topic of §2.1, suppose  $I \subset [n]$  indexes a nonempty subset of an arrangement  $\mathcal{A}$ , and let  $p_I: \mathbb{k}^n \rightarrow \mathbb{k}^I$  be the induced coordinate projection. Then  $p_I$  restricts to a map of reciprocal planes,  $p_I|_{Y(\mathcal{A})}: Y(\mathcal{A}) \rightarrow Y(\mathcal{A}_I)$ . The ring homomorphism  $p_I^*: R(\mathcal{A}_I) \rightarrow R(\mathcal{A})$  is obviously injective, so  $p_I|_{Y(\mathcal{A})}$  is dominant.



On the other hand, for  $I \subseteq [n]$ , let  $\mathbb{k}^I \subseteq \mathbb{k}^n$  denote the coordinate subspace supported on coordinates  $I$ , and  $(\mathbb{k}^*)^I = \{x \in \mathbb{k}^n : x_i \neq 0 \Leftrightarrow i \in I\}$ .

**Definition 3.1.** For any nonempty  $I \subseteq [n]$ , let  $R_I(\mathcal{A}) := R(\mathcal{A})/(y_i : i \in [n] - I)$ . Let  $Y_I(\mathcal{A}) = \text{Spec } R_I(\mathcal{A}) = Y(\mathcal{A}) \cap \mathbb{k}^I$ , the scheme-theoretic fibre over 0 of the projection  $p_{[n]-I}|_{Y(\mathcal{A})}$ . If  $I = [n] - \{i\}$  for some  $i$ , we will write  $R_{H_i}(\mathcal{A})$  in place of  $R_{[n]-\{i\}}(\mathcal{A})$ , and call this the *relative Orlik-Terao algebra*.

If  $X$  is a flat of  $\mathcal{A}$ , Proudfoot and Speyer [PS06] showed that  $Y(\mathcal{A}_X) \cong Y_{[X]}(\mathcal{A})$ : in other words, the map  $p_X^*$  is split by the homomorphism

$$s_X(y_i) = \begin{cases} y_i & \text{if } H_i \leq X; \\ 0 & \text{otherwise,} \end{cases}$$

for  $1 \leq i \leq n$ . Moreover, they showed  $L(\mathcal{A})$  indexes a stratification of  $Y(\mathcal{A})$ , which leads to the following decomposition.

**Proposition 3.2.** *For any  $I \subset [n]$ ,*

$$Y_I(\mathcal{A}) = \bigcup_{\substack{X \in L(\mathcal{A}): \\ I \subseteq [X]}} Y_{[X]}(\mathcal{A})$$

*Proof.* If  $I \subseteq J$ , the identity map on  $\mathbb{k}^n$  restricts to a map  $Y_I \rightarrow Y_J$ , hence to

$$\bigcup_{\substack{X \in L(\mathcal{A}): \\ I \subseteq [X]}} Y_{[X]}(\mathcal{A}) \rightarrow Y_I(\mathcal{A}).$$

By [PS06], the open stratum  $Y_I^\circ(\mathcal{A}) := Y(\mathcal{A}) \cap (\mathbb{k}^*)^I$  is empty, unless  $I = [X]$  for some  $X \in L(\mathcal{A})$ , so a localization argument shows the map above is in fact an isomorphism.  $\square$

Now we restrict our attention to the case in which  $|I| = n - 1$ .

**Corollary 3.3.** *For any  $H \in \mathcal{A}$ ,*

$$Y_H(\mathcal{A}) \cong \bigcup_{\substack{X \in L(\mathcal{A}): \\ H \not\leq X}} Y(\mathcal{A}_X).$$

The two parts of Theorem 2.7 have counterparts.

**Proposition 3.4.** *For any hyperplane  $H_i \in \mathcal{A}$ , the kernel of the natural map*

$$\mathbb{k}[y_j : j \in [n], j \neq i] \rightarrow R_{H_i}(\mathcal{A})$$

*is generated by elements  $\bar{r}_c$  indexed by circuits  $c$  of  $\mathcal{A}$ :*

$$(9) \quad \bar{r}_c := \begin{cases} \sum_{j \in [c]} c_j \prod_{k \in [c] - \{j\}} y_k & \text{if } i \notin [c]; \\ \prod_{j \in [c] - \{i\}} y_j & \text{if } i \in [c]. \end{cases}$$



**Proposition 3.5.** *Let  $\mathcal{A}$  be an ordered arrangement of hyperplanes. If we have  $I = \{k, k+1, \dots, n\}$  for some integer  $k$  with  $1 \leq k \leq n$ , then with respect to lexicographic order,*

$$\operatorname{In} R_I(\mathcal{A}) \cong \mathbb{k}[y_1, \dots, y_n] / J_{\mathbf{bc}(\mathcal{A})|_I},$$

where  $J_{\mathbf{bc}(\mathcal{A})|_I}$  denotes the Stanley-Reisner ideal of the subcomplex of  $\mathbf{bc}(\mathcal{A})$  supported on vertices  $I$ .

*Proof.* Since  $R_I(\mathcal{A}) \cong \mathbb{k}[y_1, \dots, y_n] / ((y_1, \dots, y_{k-1}) + I(\mathcal{A}))$ , by Theorem 2.7 it is enough to show that

$$\operatorname{In}((y_1, \dots, y_{k-1}) + I(\mathcal{A})) = (y_1, \dots, y_{k-1}) + \operatorname{In}(I(\mathcal{A})).$$

The inclusion  $\supseteq$  is immediate. In the other direction, suppose  $f \in (y_1, \dots, y_{k-1}) + I(\mathcal{A})$ . We need to show that  $\operatorname{Lt}(f) \in (y_1, \dots, y_{k-1}) + \operatorname{In}(I(\mathcal{A}))$ , where  $\operatorname{Lt}$  denotes the leading term in lexicographic order.

Write  $f = y_1 g_1 + \dots + y_{k-1} g_{k-1} + h$  for some polynomials  $g_i, h$ , where  $h \in I(\mathcal{A})$ . If  $h = 0$ , then  $\operatorname{Lt}(f) \in (y_1, \dots, y_{k-1})$ , and our claim is clear. Otherwise,  $f$  and  $h$  have the same degree. Since the variables  $y_1, \dots, y_{k-1}$  are first in order, monomials of a fixed degree that are in  $(y_1, \dots, y_{k-1})$  come before those that are not. It follows that  $\operatorname{Lt}(f) = \operatorname{Lt}(h)$ . Since  $h \in I(\mathcal{A})$ , the claim is shown.  $\square$

Since  $\mathbf{bc}(\mathcal{A})$  is a cone over  $\mathbf{bc}_0(\mathcal{A})$  with vertex 1, by [Bry77], we obtain a relative version of Theorem 2.7.

**Corollary 3.6.** *If  $H$  is first in order in an arrangement  $\mathcal{A}$ , then*

$$\operatorname{In} R_H(\mathcal{A}) \cong \mathbb{k}[y_1, \dots, y_n] / J_{\mathbf{bc}_0(\mathcal{A})},$$

where  $J_{\mathbf{bc}_0(\mathcal{A})}$  is the Stanley-Reisner ideal of the reduced broken circuit complex.

The Hilbert series formula (1) also has a relative version. For any arrangement, the Poincaré polynomial  $\pi(\mathcal{A}, t)$  is divisible by  $1 + t$ , and the polynomial  $\pi(\mathbb{P}\mathcal{A}, t) := \pi(\mathcal{A}, t) / (1 + t)$  enumerates the Betti numbers of the projective complement  $\mathbb{P}M(\mathcal{A})$  if  $\mathcal{A}$  is a complex arrangement.

**Proposition 3.7.** *If  $H$  is a hyperplane of an arrangement  $\mathcal{A}$ , then  $R_H(\mathcal{A})$  is Cohen-Macaulay, and*

$$h(R_H(\mathcal{A}), t) = \pi(\mathbb{P}\mathcal{A}, t / (1 - t)).$$

*Proof.* Since  $R(\mathcal{A})$  is a domain, the sequence

$$(10) \quad 0 \longrightarrow R(\mathcal{A})[-1] \xrightarrow{\cdot y_i} R(\mathcal{A}) \longrightarrow R_H(\mathcal{A}) \longrightarrow 0$$

is exact, where  $H = H_i$ . The first result then follows from the Cohen-Macaulay property for  $R(\mathcal{A})$  from [PS06], and the second from (1).  $\square$

**Example 3.8.** Suppose  $\mathcal{A}$  is an arrangement of rank 2. Every 3-element set  $C \subseteq 2, \dots, n$  is a circuit, and

$$R_{H_1}(\mathcal{A}) \cong \mathbb{k}[y_2, \dots, y_n] / (y_i y_j : 2 \leq i < j \leq n),$$

using (9) from Proposition 3.4, and

$$h(R_{H_1}(\mathcal{A}), t) = 1 + (n-1)t / (1-t). \quad \diamond$$

**3.2. Coordinate intersections and modular flats.** In this section, we compare the schemes  $Y_{[n]-[X]}(\mathcal{A})$  and  $Y_X(\mathcal{A}_{v,X})$ . Recall  $X$  is a hyperplane of the fibre arrangement  $\mathcal{A}_{v,X}$  (Definition 2.1): the ambient affine space of both schemes, then, may be identified with  $\mathbb{k}^{[n]-[X]}$ . Now we show that, for generic  $v$ , we have  $Y_{[n]-[X]}(\mathcal{A}) \cong Y_X(\mathcal{A}_{v,X})$  if and only if  $X$  is a modular flat. Our comparison has the following purely combinatorial foundation. By reordering the hyperplanes of  $\mathcal{A}$  if necessary, we will often assume that  $[X] = \{1, 2, \dots, n_X\}$ , a condition which we shall abbreviate by saying  $X$  is an *initial flat* of  $\mathcal{A}$ .

**Theorem 3.9.** *Suppose  $X$  is an initial flat of  $\mathcal{A}$ . Then  $\mathbf{bc}_0(\overline{T}_X(\mathcal{A}))$  is a subcomplex of  $\mathbf{bc}(\mathcal{A})|_{[n]-[X]}$ . The two complexes are equal if and only if  $X$  is modular.*

Before giving the proof, we recall two facts about modular flats. The first is a description of the lattice  $L(\overline{T}_X(\mathcal{A}))$  in the modular case.

**Lemma 3.10** (Prop. 5.14(3), [Bry75], Prop. 2.3, [FP02]). *If  $X \in L(\mathcal{A})$  is modular,*

$$L(\overline{T}_X(\mathcal{A})) \cong \{Y \in L(\mathcal{A}) : X \wedge Y = V \text{ or } X \leq Y\}.$$

*Moreover, the rank function is given by*

$$\rho_{\overline{T}_X(\mathcal{A})}(Y) = \begin{cases} \rho(Y) & \text{if } X \wedge Y = V; \\ \rho(Y) - \rho(X) + 1 & \text{if } X \leq Y. \end{cases}$$

Second, Brylawski's "short-circuit axiom" of [Bry75] characterizes modularity in terms of circuits:  $X$  is modular if and only if, for all  $C \in \mathcal{C}(\mathbf{M}(\mathcal{A}))$  for which  $C - [X] \neq \emptyset$ , there exists some  $q \in [X]$  such that  $(C - [X]) \cup \{q\}$  is a dependent set. To reformulate slightly, let  $\partial(S) := \text{cl}(S) - S$ , for any  $S \subseteq [n]$ . Then the short-circuit axiom can also be stated as

**Lemma 3.11.** *An initial flat  $X$  of  $\mathbf{M}(\mathcal{A})$  is modular if and only if, for all  $C \in \mathcal{C}(\mathbf{M}(\mathcal{A}))$ , either  $C \subseteq [X]$ ,  $C \subseteq [n] - [X]$ , or  $\partial(C - [X])$  is nonempty and its least element is in  $[X]$ .*

It will be convenient to isolate a technical lemma:

**Lemma 3.12.** *Suppose that  $X \in L(\mathcal{A})$  is initial, and  $\mathbf{bc}_0(\overline{T}_X(\mathcal{A})) = \mathbf{bc}(\mathcal{A})|_{[n]-[X]}$ . For any  $C \in \mathcal{C}(\mathbf{M}(\mathcal{A}))$  with  $C \not\subseteq [X]$ , the set  $C - [X]$  is a broken circuit of  $\mathbf{M}(\mathcal{A})$ .*

*Proof.* If  $C \cap [X] = \emptyset$ , the claim is obvious; otherwise,  $(C - [X]) \cup \{0\}$  is a circuit of  $\overline{T}_X(\mathcal{A})$ . Since  $0 < \min(C - [X])$ , we see  $C - [X]$  is a broken circuit of  $\overline{T}_X(\mathcal{A})$ . By hypothesis, then,  $C - [X]$  is also a broken circuit of  $\mathcal{A}$ .  $\square$

*Proof of Theorem 3.9.* Fix  $\mathcal{A}$  and  $X$ , and let  $\Delta = \mathbf{bc}(\mathcal{A})|_{[n]-[X]}$  and  $\Delta_0 = \mathbf{bc}_0(\overline{T}_X(\mathcal{A}))$  for short. First, we show that  $\Delta_0 \subseteq \Delta$  for all  $X$ .

If  $\sigma \in \Delta_0$ , suppose  $\sigma \notin \Delta$ . Since the set  $\sigma$  is independent in  $\mathbf{M}(\mathcal{A})$ , by choosing  $\sigma$  to be minimal, we may assume  $\sigma$  is a broken circuit: that is, there exists some  $i$  for which  $\sigma \cup \{i\} \in \mathcal{C}(\mathbf{M}(\mathcal{A}))$  and  $i < \min(\sigma)$ . We must have  $i \leq n_X$ , since otherwise  $\sigma \cup \{i\}$  would be a circuit in  $\overline{T}_X(\mathcal{A})$ , contrary to assumption. But then  $\sigma \cup \{0\}$  is dependent in  $\overline{T}_X(\mathcal{A})$ , again a contradiction. It follows that  $\Delta_0 \subseteq \Delta$ .

Now suppose  $X$  is modular. We assume  $\rho(X) > 1$ , since the case where  $X$  is a hyperplane is trivial. Suppose that  $S \subseteq [n] - [X]$  is a minimal non-face of  $\Delta_0$ . That is,  $S$  is a broken circuit, meaning  $S \cup \{q\}$  is a circuit of  $\overline{T}_X(\mathcal{A})$  for some  $q < \min(S)$ . First, suppose  $q = 0$ . By Lemma 3.10, we must have  $X \wedge \text{cl}(S) \neq V$ , so there exists some  $q \in [X] \cap \text{cl}(S)$ . That is,  $S \cup \{q\}$  is dependent in  $\mathbf{M}(\mathcal{A})$ , so there is a circuit  $C \subseteq S \cup \{q\}$ . Then  $C - \{q\} \cup \{0\}$  is a dependent set in  $\overline{T}_X(\mathcal{A})$ : by assumption on  $S$ , we must have  $C = S \cup \{q\}$ . Since  $q \leq n_X < \min(S)$ , we see  $S$  is a broken circuit of  $\mathbf{M}(\mathcal{A})$ , and a non-face of  $\Delta$ .

Otherwise,  $q > n_X$ . If  $S \cup \{q\}$  is a circuit of  $\mathbf{M}(\mathcal{A})$ , again we are done, so assume  $S \cup \{q\}$  is independent in  $\mathbf{M}(\mathcal{A})$ . Let  $Y_0 = \text{cl}(S)$  and  $Y = \text{cl}(S \cup \{q\})$  in  $L(\mathcal{A})$ . Then  $\rho(Y) = |S| + 1 > \rho_{\overline{T}_X(\mathcal{A})}(Y)$ , so by Lemma 3.10, we must have  $X \leq Y$ . Since  $X$  is modular,

$$\begin{aligned} \rho(X \wedge Y_0) &= \rho(X) + \rho(Y_0) - \rho(Y) \\ &= \rho(X) - 1 \\ &\geq 1, \end{aligned}$$

by hypothesis. It follows that  $S \cup \{r\}$  is dependent for some  $r \in [X]$ . Since then  $r < \min(S)$ , we see  $S$  is again a non-face of  $\Delta$ .

Finally, we check the converse. Suppose that  $\Delta_0 = \Delta$ , and we show  $X$  is modular using Brylawski's "short circuit axiom" (Lemma 3.11). If  $C \not\subseteq [X]$ , then  $\partial(C - [X])$  is nonempty by Lemma 3.12, so we must show  $\min \partial(C - [X]) \in [X]$ .

Order subsets  $S, T \subseteq [n]$  so that  $S \prec T$  if either  $|S| < |T|$  or  $|S| = |T|$  and  $S$  precedes  $T$  in lexicographic order. Let

$$\mathcal{C}_X = \{C \in \mathcal{C}(\mathbf{M}(\mathcal{A})) : C \not\subseteq [X] \text{ and } \min \partial(C - [X]) \notin [X]\}.$$

If  $X$  is not modular, then  $\mathcal{C}_X$  is not empty, and it contains a minimal element  $C_1$ . Let  $q = \min \partial(C_1 - [X])$ : by assumption,  $q > n_X$ . Then, for some  $S \subseteq C_1 - [X]$ , we have a circuit  $S \cup \{q\}$  with  $q < \min S$ . Let  $r = \max S$ . By the circuit exchange axiom, there exists a circuit

$$\begin{aligned} C_2 &\subseteq C_1 \cup S \cup \{q\} - \{r\} \\ &= C_1 \cup \{q\} - \{r\}. \end{aligned}$$

By construction,  $q < r$ , so  $C_2 \prec C_1$ . If  $C_2 \subseteq [X]$ , note that  $C_2 - \{q\} \subseteq C_1 - [X]$ , so circuit exchange using  $C_2$  and  $S \cup \{q\}$  to eliminate  $q$  would give a circuit contained in  $C_1 - [X]$ , a contradiction. So  $C_2 \not\subseteq [X]$ .

Let  $p = \min \partial(C_2 - [X])$ . Then  $T \cup \{p\}$  is a circuit of  $\mathbf{M}(\mathcal{A})$  for some broken circuit  $T \subseteq C_2 - [X]$ . By minimality of  $C_1$  in  $\mathcal{C}_X$ , we have  $p \in [X]$ , so  $p < q$ . Note that  $q \in T$ , since if not,  $T \subseteq C_1 - [X]$ . But then  $p \in \partial(C_1 - [X])$ , contradicting our choice of  $q$  (since  $p < q$ ).

To complete the argument, we use the circuit exchange axiom again with  $S \cup \{q\}$  and  $T \cup \{p\}$  to obtain a circuit

$$\begin{aligned} C_3 &\subseteq S \cup \{q\} \cup T \cup \{p\} - \{q\} \\ &\subseteq (C_1 - [X]) \cup \{p\}. \end{aligned}$$

Clearly  $p \in C_3$ , which contradicts the choice of  $q$ . We conclude that  $\mathcal{C}_X$  is empty.  $\square$

This leads to a comparison of coordinate intersections of reciprocal planes.

**Theorem 3.13.** *If  $X$  is a flat of an arrangement  $\mathcal{A}$ , then for any closed point  $v$  in  $\mathbb{P}M(\mathcal{A}_X)$ , there is a surjection of algebras*

$$(11) \quad p: R_{[n]-[X]}(\mathcal{A}) \rightarrow R_X(\mathcal{A}_v),$$

*The map  $p$  is an isomorphism if and only if  $X$  is a modular flat.*

*Proof.* Let  $\phi: Y_X(\mathcal{A}_v) \rightarrow \mathbb{k}^{[n]-[X]}$  denote the natural embedding. We claim that  $\phi^*$  factors through  $R_{[n]-[X]}(\mathcal{A})$ . To see this, using (6) and Definition 3.1, we must check that the ideal  $I(\mathcal{A}) + (y_i: i \in [X])$  maps to zero. Since relations  $r_c$  from (7) indexed by circuits  $c$  of  $\mathcal{A}$  generate  $I(\mathcal{A})$ , it is enough to show  $\phi^*(r_c) = 0$  for all  $c$ , and we do so by considering three cases.

- (1)  $|[c] \cap [X]| \geq 2$ : in this case,  $r_c$  is zero in  $R_{[n]-[X]}(\mathcal{A})$ , since each monomial contains a variable indexed by  $[X]$ .
- (2)  $[c] \cap [X] = \{i\}$ , for some  $i$ : then  $\tilde{C} := \{0\} \cup [c] - \{i\}$  is a circuit in  $\overline{T}_X(\mathcal{M}(\mathcal{A}))$ . The image of the element  $r_c$  in  $R_X(\mathcal{A})$  is a unit multiple of  $\prod_{j \in [c] - \{i\}} y_j$  by (7). Then  $\phi^*(r_c)$  is zero, in view of the relation indexed by  $\tilde{C}$  in (9).
- (3)  $[c] \cap [X] = \emptyset$ : then  $c$  is also a circuit of  $\mathcal{A}_v$ , so the image of  $r_c$  in  $R_X(\mathcal{A}_v)$  is zero.

Since  $\phi^*$  is surjective, so is the induced map  $p$  of (11).

To prove the second claim, we order the hyperplanes of  $\mathcal{A}$  so that  $[X]$  is initial, and pass to initial ideals. Using Proposition 3.5, we see that  $p$  induces a map of Stanley-Reisner rings

$$\text{In}(p): \mathbb{k}[y_1, \dots, y_n]/J_{\mathbf{bc}(\mathcal{A})_{[n]-[X]}} \rightarrow \mathbb{k}[y_0, y_k, y_{k+1}, \dots, y_n]/J_{\mathbf{bc}_0(\mathcal{A}_{v,X})}.$$

Since the map is the identity on the nonzero degree-1 elements, it induces a map of simplicial complexes. If  $X$  is modular, then,  $\text{In}(p)$  is an isomorphism, by Theorem 3.9, then  $p$  is as well. Conversely, if  $p$  is an isomorphism, then  $\mathbf{bc}(\mathcal{A})_{[n]-[X]} \cong \mathbf{bc}_0(\mathcal{A}_v)$ . It follows that  $\mathcal{M}(\mathcal{A}_v) = \overline{T}_X(\mathcal{M}(\mathcal{A}))$ , using Proposition 2.3: if not,  $\mathcal{M}(\mathcal{A}_v)$  has strictly more dependent sets than  $\overline{T}_X(\mathcal{M}(\mathcal{A}))$ , so  $\mathbf{bc}_0(\mathcal{A}_v)$  would be a proper subcomplex of  $\mathbf{bc}_0(\overline{T}_X(\mathcal{A}))$ , contradicting Theorem 3.9.  $\square$

It follows from Theorem 3.13 that:

**Corollary 3.14.** *If  $X$  is modular, the algebras  $R_X(\mathcal{A}_v)$  are isomorphic, for all closed points  $v \in \mathbb{P}M(\mathcal{A}_X)$ .*

**3.3. An algebra decomposition.** Now we connect the results above. Let  $X$  be a flat of an arrangement  $\mathcal{A}$ , and let  $\pi: R(\mathcal{A}) \rightarrow R_{[n]-[X]}(\mathcal{A})$  be the natural surjection. Let  $v \in \mathbb{P}M(\mathcal{A})$  be generic. If  $C \in \mathbf{bc}_0(\overline{T}_X(\mathcal{A}))$ , then  $C$  is also a simplex of  $\mathbf{bc}(\mathcal{M}(\mathcal{A}))$ , by Theorem 3.9. By Corollary 3.6, then, there is an additive map  $\iota: R_X(\mathcal{A}_v) \rightarrow R(\mathcal{A})$  given by inclusion of monomials supported on broken circuits. By construction,  $\iota$  is

a section of the surjection  $p \circ \pi: R(\mathcal{A}) \rightarrow R_X(\mathcal{A}_v)$  from Theorem 3.13. We define a map of  $R(\mathcal{A}_X)$ -modules,

$$(12) \quad \tau_X: R(\mathcal{A}_X) \otimes_{\mathbb{k}} R_X(\mathcal{A}_v) \rightarrow R(\mathcal{A})$$

by setting  $\tau_X(x \otimes z) = x \cdot \iota(z)$ , for  $x \in R(\mathcal{A}_X)$  and  $z \in R_X(\mathcal{A}_v)$ . This map turns out to be most interesting when  $X$  is modular.

**Theorem 3.15.**  *$X$  is a modular flat of  $\mathcal{A}$  if and only if  $\tau_X$  is an isomorphism of  $R(\mathcal{A}_X)$ -modules:  $R(\mathcal{A}) \cong R(\mathcal{A}_X) \otimes_{\mathbb{k}} R_X(\mathcal{A}_v)$ .*

*Proof.* Without loss, assume  $X$  is initial. Suppose  $X$  is modular: then the broken circuit complex decomposes as a join, by [BO81, Thm. 1.6].

$$\begin{aligned} \mathbf{bc}(\mathcal{A}) &\cong \mathbf{bc}(\mathcal{A}_X) * \mathbf{bc}(\mathcal{A})|_{[n]-[X]}, \\ &\cong \mathbf{bc}(\mathcal{A})_{[X]} * \mathbf{bc}_0(\overline{T}_X(\mathcal{A})), \end{aligned}$$

by Theorem 3.9. Using Theorem 2.7 and isomorphism (8),

$$\begin{aligned} \text{In } R(\mathcal{A}) &\cong \mathbb{k}[y_1, \dots, y_n] / J_{\mathbf{bc}(\mathcal{A})} \\ &\cong \mathbb{k}[y_1, \dots, y_n] / (J_{\mathbf{bc}(\mathcal{A}_X)} + J_{\mathbf{bc}_0(\overline{T}_X(\mathcal{A}))}), \\ (13) \quad &\cong \text{In } R(\mathcal{A}_X) \otimes_{\mathbb{k}} \text{In } R_X(\mathcal{A}_v), \end{aligned}$$

by (8) again and Corollary 3.6.

If  $y \in R(\mathcal{A})$  is a monomial supported on a broken circuit, then, we have  $y = xz$  for such monomials  $x \in \text{In } R(\mathcal{A}_X)$  and  $z \in \text{In } R_X(\mathcal{A}_v)$ . By construction,  $y = \tau_X(x \otimes z)$ , so  $\tau_X$  is surjective. On the other hand, (13) shows the domain and codomain are additively isomorphic, so  $\tau_X$  is an isomorphism of  $R(\mathcal{A}_X)$ -modules.

Conversely, if  $\tau_X$  is an isomorphism, so is (13), in which case  $\mathbf{bc}(\mathcal{A})$  decomposes as a join of  $\mathbf{bc}(\mathcal{A})_{[X]}$  with another complex, in which case  $X$  is modular by [BO81, Thm. 1.6].  $\square$

**Corollary 3.16.** *If  $X$  is a modular flat of  $\mathcal{A}$ , then  $R(\mathcal{A})$  is a free module over  $R(\mathcal{A}_X)$ .*

Using the map (11), there is also a  $R(\mathcal{A}_X)$ -module map

$$(14) \quad \tau_X \circ (1 \otimes p): R(\mathcal{A}_X) \otimes_{\mathbb{k}} R_{[n]-[X]}(\mathcal{A}) \rightarrow R(\mathcal{A}).$$

Combining Theorems 3.13 and 3.15, we obtain the following.

**Corollary 3.17.**  *$X$  is a modular flat of  $\mathcal{A}$  if and only if the map (14) is an isomorphism.*

*Proof.* If  $X$  is modular, then combining Theorems 3.13 and 3.15 shows (14) is an isomorphism. Conversely, suppose  $\tau_X$  is an isomorphism. Ordering the hyperplanes of  $\mathcal{A}$  so that the indices  $[X]$  come first, we obtain an isomorphism

$$\text{In}(R(\mathcal{A})) \cong \text{In}(R(\mathcal{A}_X)) \otimes_{\mathbb{k}} \text{In}(R_{[n]-[X]}(\mathcal{A}))$$

By Proposition 3.5, it follows that the broken circuit complex decomposes as a join,  $\mathbf{bc}(\mathcal{A}) \cong \mathbf{bc}(\mathcal{A})|_{[X]} * \mathbf{bc}(\mathcal{A})|_{[n]-[X]}$ . By [BO81, Thm. 11],  $X$  is modular.  $\square$

We note that the Orlik-Solomon algebra analogue of this result appears as [BZ91, Cor. 5.5].

## 4. APPLICATIONS

The Modular Fibration Theorem 2.5 was first formulated in the case of  $X$  a coatom, since then the fibre is 1-dimensional. Similarly, our algebra decomposition (12) can be refined in this case.

**4.1. The coatomic case.** Suppose that  $X$  is a modular coatom of an arrangement  $\mathcal{A}$ . As usual, we order the hyperplanes of  $\mathcal{A}$  so that  $\mathcal{A}_X = \{H_1, \dots, H_{n_X}\}$ . For any  $i, j$  satisfying  $n_X < i < j \leq n$ , since  $X$  is modular, we have

$$X + H_i \cap H_j = H, \quad \text{for some (unique) hyperplane } H \in \mathcal{A}_X.$$

Let  $i \circ j$  denote the integer in  $[n_X]$  for which  $H = H_{i \circ j}$ . Since  $\{i, j, i \circ j\}$  is a circuit, we have  $f_{i \circ j} = a_{ij}f_i + b_{ij}f_j$ , for some scalars  $a_{ij}$  and  $b_{ij}$ , for all  $n_X < i < j \leq n$ .

**Theorem 4.1.** *Suppose  $X$  is a modular coatom of  $\mathcal{A}$ . There is an algebra isomorphism*

$$R(\mathcal{A}) \cong R(\mathcal{A}_X)[z_i : n_X < i \leq n]/J,$$

where  $J$  is the ideal generated by  $z_i z_j - y_{i \circ j}(a_{ij} z_j + b_{ij} z_i)$ , for all  $n_X < i < j \leq n$ .

*Proof.* By Theorem 3.15, as  $R(\mathcal{A}_X)$ -modules,

$$\begin{aligned} R(\mathcal{A}) &\cong R(\mathcal{A}_X) \otimes_{\mathbb{k}} R_X(\mathcal{A}_v) \\ &\cong R(\mathcal{A}_X) \otimes_{\mathbb{k}} \mathbb{k}[z_i : n_X < i \leq n]/(z_i z_j : n_X < i < j \leq n), \end{aligned}$$

using Example 3.8. To specify the structure of  $R(\mathcal{A})$  as an  $R(\mathcal{A}_X)$ -algebra, it is enough to observe that the relations  $r_C$  from (7) for the circuits  $C = \{i, j, i \circ j\}$  generate the kernel of the natural map  $R(\mathcal{A}_X)[z_i : n_X < i \leq n] \rightarrow R(\mathcal{A})$ .  $\square$

**4.2. Tor algebras and the Koszul property.** If  $X$  is a flat of  $\mathcal{A}$ , we consider the Eilenberg-Moore spectral sequence of the fibre sequence

$$R(\mathcal{A}_X) \hookrightarrow R(\mathcal{A}) \twoheadrightarrow R_{[n]-[X]}(\mathcal{A}).$$

If  $X$  is modular, by Corollary 3.16,  $R(\mathcal{A})$  is a free  $R(\mathcal{A}_X)$ -module. So by [CE99, XVI.6.1], we have

$$(15) \quad E_{pq}^2 = \text{Tor}_p^{R_{[n]-[X]}(\mathcal{A})}(\text{Tor}_q^{R(\mathcal{A}_X)}(A, \mathbb{k}), B) \Rightarrow \text{Tor}_{p+q}^{R(\mathcal{A})}(A, B)$$

for any  $R(\mathcal{A}_X)$ -module  $A$  and  $R(\mathcal{A})$ -module  $B$ . Recall that a standard graded  $\mathbb{k}$ -algebra  $A$  is *Koszul* if  $\text{Tor}_p^A(\mathbb{k}, \mathbb{k})_q = 0$  for  $p \neq q$ .

**Lemma 4.2.** *Let  $H$  be a hyperplane of an arrangement  $\mathcal{A}$ . Then  $R(\mathcal{A})$  is Koszul if and only if  $R_H(\mathcal{A})$  is Koszul.*

*Proof.* From the short exact sequence (10), the map  $R(\mathcal{A}) \rightarrow R_H(\mathcal{A})$  is Koszul, and the claim follows by [PP05, §2.5, Example 1].  $\square$

**Theorem 4.3.** *Suppose that  $X$  is a modular flat of  $\mathcal{A}$ . If  $R_X(\mathcal{A}_v)$  and  $R(\mathcal{A}_X)$  are Koszul, then so is  $R(\mathcal{A})$ .*

*Proof.* By Theorem 3.13, we have  $R_{[n]-[X]}(\mathcal{A}) \cong R_X(\mathcal{A}_v)$ , which is Koszul if and only if  $R(\mathcal{A}_v)$  is, by Lemma 4.2. The claim follows directly by examining the grading in (15), taking  $A = C = \mathbb{k}$ .  $\square$

For rank-2 arrangements, we may compute directly.

**Lemma 4.4.** *If  $\mathcal{A}$  is an arrangement of rank  $\ell \leq 2$ , then  $R(\mathcal{A})$  is Koszul.*

*Proof.* If  $\ell = 1$ , the claim is immediate. If  $\ell = 2$ , by Example 3.8, for any  $H \in \mathcal{A}$ , the algebra  $R_H(\mathcal{A})$  is a quotient of a polynomial ring by quadratic monomials. It follows  $R_H(\mathcal{A})$  is Koszul, by [Ani86], and so is  $R(\mathcal{A})$ , by Lemma 4.2.  $\square$

Note that combining the last two statements gives an inductive proof of Theorem 2.8.

**Example 4.5.** The  $X_2$  arrangement is defined by  $Q = xyz(x+y)(x-z)(y-z)(x+y-2z)$ . A computation using Macaulay 2 [GS] shows that the Orlik-Terao algebra is quadratic, but not Koszul.  $\diamond$

An arrangement is said to be *2-formal* if the vector space of linear relations amongst the polynomials  $\{f_i\}_{i=1}^n$  are generated by those from circuits of size three: see [ST09] for a full discussion of this property.

**Corollary 4.6.** *For any arrangement  $\mathcal{A}$  we have*

$$\begin{aligned} \mathcal{A} \text{ is supersolvable} &\Rightarrow R(\mathcal{A}) \text{ is a } G\text{-algebra} \Rightarrow \\ R(\mathcal{A}) \text{ is Koszul} &\Rightarrow R(\mathcal{A}) \text{ is quadratic} \Rightarrow \mathcal{A} \text{ is 2-formal.} \end{aligned}$$

*The last three implications are strict.*

*Proof.* The two implications come from Theorem 2.8. The next is immediate, and Example 4.5 shows it is not reversible. For the last implication, let  $I_2(\mathcal{A})$  denote the ideal generated by the degree-2 elements of  $I(\mathcal{A})$ . Then if  $I(\mathcal{A})$  is quadratic,  $\text{codim}(I_2(\mathcal{A})) = \text{codim}(I(\mathcal{A})) = n - \ell$ . By [ST09, Theorem 2.4], this is equivalent to  $\mathcal{A}$  being 2-formal.

However, the converse is not true: consider the non-Fano arrangement  $\mathcal{A}$  ([ST09], Example 1.7), defined by  $Q(\mathcal{A}) = xyz(x-y)(x-z)(y-z)(x+y-z)$ . Although  $\mathcal{A}$  is 2-formal,  $I(\mathcal{A})$  is not quadratic.  $\square$

**Question 4.7.** Do there exist arrangements  $\mathcal{A}$  for which  $R(\mathcal{A})$  is Koszul, yet  $\mathcal{A}$  is not supersolvable? The analogous problem is also open for cohomology rings ([SY97, §5].)  $\diamond$

**4.3. Restrictions and resolutions.** Even in some cases for which the Orlik-Terao algebra is not Koszul, it is still possible to describe part of the algebra  $\text{Tor}^{R(\mathcal{A})}(\mathbb{k}, \mathbb{k})$  explicitly.

Suppose that  $W \subseteq V$  are both linear subspaces of  $\mathbb{k}^n$  and  $V \not\subseteq \hat{H}_i$  for  $1 \leq i \leq n$ , as in §1.1. If  $\mathcal{A} = \{\hat{H}_i \cap V : i \in [n]\}$  as before, let  $I = \{i \in [n] : W \not\subseteq \hat{H}_i\}$ . arrangement  $\mathcal{A}^W := \{\hat{H}_i \cap W : i \in I\}$  is called the *restriction of  $\mathcal{A}$  to  $W$* . Let  $M(\mathcal{A}^W) = W \cap (\mathbb{k}^*)^I$ , the complement of the hyperplanes  $\mathcal{A}^W$ . We say that the restriction  $\mathcal{A}^W$  is *k-generic* if  $L_p(\mathcal{A}^W) \cong L_p(\mathcal{A})$  for  $p \leq k$ . We note that the condition that  $\mathcal{A}^W$  is 1-generic is equivalent to  $I = [n]$ , in which case  $Y(\mathcal{A}^W)$  is a subscheme of  $Y(\mathcal{A})$ .



**Proposition 4.8.** *If  $\mathcal{A}^W$  is a 1-generic restriction, then the map  $R(\mathcal{A}) \rightarrow R(\mathcal{A}^W)$  is surjective. The kernel is*

$$I_{\mathcal{A}}(\mathcal{A}^W) := (\{r_c : c \text{ is a circuit of } \mathcal{A}^W \text{ and not of } \mathcal{A}\}).$$

*If, moreover,  $\mathcal{A}^W$  is a  $p$ -generic restriction for  $p \geq 1$ , then  $I_{\mathcal{A}}(\mathcal{A}^W)$  is generated in degrees strictly greater than  $p$ .*

*Proof.* Follows immediately from Theorem 2.7.  $\square$

If  $\mathcal{A}^W$  is a 2-generic restriction, then any circuits of  $\mathcal{A}^W$  which are not circuits of  $\mathcal{A}$  have at least 4 elements. It follows that the ideal  $I_{\mathcal{A}}(\mathcal{A}^W)$  is zero in degrees  $\leq 2$ . Since  $\text{Tor}_p^{R(\mathcal{A})}(\mathbb{k}, \mathbb{k})_p$ , depends only on the degree  $\leq 2$  part of  $R(\mathcal{A})$ , for  $p \geq 0$ , we have

$$\text{Tor}_p^{R(\mathcal{A}^W)}(\mathbb{k}, \mathbb{k})_p \cong \text{Tor}_p^{R(\mathcal{A})}(\mathbb{k}, \mathbb{k})_p,$$

for all  $p \geq 0$ : equivalently, the quadratic dual algebras are isomorphic,  $R(\mathcal{A})^\dagger \cong R(\mathcal{A}^W)^\dagger$ .

The family of hypersolvable arrangements, introduced by Jambu and Papadima in [JP98], interpolates between supersolvable and generic arrangements. Like the former, they are defined recursively; however, we use an equivalent characterization from [JP02], summarized in [DS06, Thm. 4.2]. An arrangement  $\mathcal{A}$  is *hypersolvable* if  $\mathcal{A} = \mathcal{B}^W$ , where  $\mathcal{B}$  is supersolvable, and  $\mathcal{B}^W$  is a 2-generic restriction. If  $\mathcal{A}$  is hypersolvable, then, the linear strand of the resolution of  $\mathbb{k}$  over  $R(\mathcal{A})$  or, equivalently,  $\text{Tor}_p^{R(\mathcal{A})}(\mathbb{k}, \mathbb{k})_p$  for  $p \geq 0$ , agrees with that of the supersolvable arrangement  $\mathcal{B}$ . From Proposition 4.8,  $R(\mathcal{A}) \cong R(\mathcal{B})/I_{\mathcal{B}}(\mathcal{A})$ , where  $I_{\mathcal{B}}(\mathcal{A})$  is generated in degrees 3 and higher.

Recall that an arrangement  $\mathcal{A}$  is called *generic* if it is an  $\ell - 1$ -generic restriction of a Boolean arrangement, where  $\ell$  is the rank of  $\mathcal{A}$ . Such an arrangement is hypersolvable, provided that  $\ell \geq 3$ : in this case, the supersolvable counterpart is the Boolean arrangement,  $R(\mathcal{B}) = \mathbb{k}[y_1, \dots, y_n]$ , and  $I_{\mathcal{B}}(\mathcal{A}) = I(\mathcal{A})$  is generated in degree  $\ell$ . Schenck showed in [Sch11, Thm. 3.7] that the regularity of  $R(\mathcal{A})$  for any arrangement  $\mathcal{A}$  is bounded above by  $\ell - 1$ : although he states the result for  $\mathbb{k} = \mathbb{C}$ , the same argument clearly works in general. In this case, it follows that  $I(\mathcal{A})$  has a linear resolution, so the natural map

$$\mathbb{C}[y_1, \dots, y_n] \twoheadrightarrow R(\mathcal{A})$$

is Golod [BF85]. Using this, we are able to compute the Betti-Poincaré series of  $\text{Tor}_p^{R(\mathcal{A})}(\mathbb{k}, \mathbb{k})$  for all generic arrangements, and we do so in the next section.

For more general hypersolvable arrangements, we lack information about the regularity of the ideal  $I_{\mathcal{B}}(\mathcal{A})$ , which prevents us from carrying out a similar analysis to the one in [DS06] for Tor algebras of Orlik-Solomon algebras of hypersolvable arrangements. By analogy, it seems reasonable to expect that  $R(\mathcal{A})$  is  $(\ell - 1)$ -regular over  $R(\mathcal{B})$ .

**4.4. Generic arrangements.** By Proposition 2.6,  $I(\mathcal{A})$  contains the  $\ell$ th fitting ideal of a  $(n-1) \times \ell$  matrix  $A_0$  over  $\mathbb{k}[y_1, \dots, y_n]$ , provided that  $n > \ell$ . For  $n > \ell \geq 3$ , let  $\mathcal{A}_{n,\ell}$  denote a generic arrangement of  $n$  hyperplanes of rank  $\ell$ . The only circuits of  $\mathcal{A}_{n,\ell}$  have  $\ell+1$  elements, so in fact we have an equality,  $I(\mathcal{A}_{n,\ell}) = \text{Fitt}_\ell(A_0)$ . Since  $I(\mathcal{A}_{n,\ell})$  has codimension  $n-\ell$ , it has an Eagon-Northcott resolution: for details, we refer to [Eis95, §A2.6]. This was observed first in the case  $\ell = 3$  in [ST09].

Let  $S = R(\mathcal{B}) = \mathbb{k}[y_1, \dots, y_n]$ , and let

$$Q_{n,\ell}(t) = \sum_{p \geq 0} \dim_{\mathbb{k}} \text{Tor}_p^S(I(\mathcal{A}_{n,\ell}), \mathbb{k})_{p+\ell} t^p.$$

An Eagon-Northcott resolution gives rise to Betti numbers

$$(16) \quad Q_{n,\ell}(t) = \sum_{p=0}^{n-\ell-1} \binom{n-1}{\ell+p} \binom{\ell-1+p}{\ell-1} t^p.$$

Let

$$P_{n,\ell}(s, t) = \sum_{p, q \geq 0} \dim_{\mathbb{k}} (\text{Tor}_p^{R(\mathcal{A}_{n,\ell})}(\mathbb{k}, \mathbb{k})_q) s^p t^q$$

denote the Poincaré-Betti series of  $R(\mathcal{A}_{n,\ell})$ . Then the Golod property implies the following.

**Corollary 4.9.** *If  $\mathcal{A}_{n,\ell}$  is a generic arrangement and  $n > \ell \geq 3$ , the Betti-Poincaré series of  $\text{Tor}^{R(\mathcal{A})}(\mathbb{k}, \mathbb{k})$ , is given by*

$$P_{n,\ell}(s, t) = \frac{(1+st)^n}{1-s^2 t^\ell Q_{n,\ell}(st)}.$$

While it is not clear that this formula admits any interesting simplification, we remark that the Betti numbers of the ideals  $I(\mathcal{A}_{n,\ell})$  assemble together into a simple generating function.

**Proposition 4.10.** *For all  $n \geq \ell \geq 3$ ,  $Q_{n,\ell}(t)$  is the coefficient of  $x^\ell y^n$  in the formal power series*

$$\frac{y}{1-y} \cdot \frac{1-(1+t)y}{1-(1+t+x)y}.$$

*Proof.* Use binomial expansions to simplify:

$$\begin{aligned} \sum_{n, \ell, p \geq 0} \binom{n-1}{\ell+p} \binom{\ell-1+p}{\ell-1} x^\ell y^n t^p &= \sum_{\ell, p \geq 0} \frac{y^{\ell+1+p}}{(1-y)^{\ell+1+p}} \binom{\ell-1+p}{\ell-1} t^p x^\ell \\ &= \frac{y}{1-y} \sum_{\ell \geq 0} \left( \frac{xy}{1-y} \right)^\ell \frac{1}{(1-ty/(1-y))^\ell} \\ &= \frac{y}{1-y} \cdot \frac{1-(1+t)y}{1-(1+t+x)y}. \quad \square \end{aligned}$$

## 5. COMPLETE INTERSECTIONS

It is known in general that a quadratic algebra that is a complete intersection is Koszul (see [PP05], Section 2.6, Example 2). Since the Orlik-Terao ideal  $I(\mathcal{A})$  has codimension  $n - \ell$ , the algebra  $R(\mathcal{A})$  is Koszul if  $I(\mathcal{A})$  is generated by  $n - \ell$  quadrics. This is clearly a rather special situation, and we characterize the arrangements for which this happens. We show (Theorem 5.11) that  $I(\mathcal{A})$  is a quadratic complete intersection (“q.c.i.”) if and only if  $\mathcal{A}$  is supersolvable with exponents 1 and 2. (So, if  $R(\mathcal{A})$  is a Koszul complete intersection, then  $\mathcal{A}$  is supersolvable: see Question 4.7.)

**5.1. A numerical constraint.** Recall that for  $X \in L_2(\mathcal{A})$ , the number of hyperplanes containing  $X$  is  $1 - \mu_{\mathcal{A}}(X)$ . If  $\mu_{\mathcal{A}}(X) = -2$ , we will call  $X$  a *triple point*.

**Theorem 5.1.** *Let  $\mathcal{A}$  be an arrangement of  $n$  hyperplanes, of rank  $\ell$  such that  $R(\mathcal{A})$  is quadratic. Then the following are equivalent:*

- (1)  $R(\mathcal{A})$  is a complete intersection.
- (2) For all  $X \in L_2(\mathcal{A})$ , we have  $|\mu_{\mathcal{A}}(X)| \leq 2$ , and the number of triple points equals  $n - \ell$ .
- (3)  $\pi(\mathcal{A}, t) = (1 + t)^{2\ell - n}(1 + 2t)^{n - \ell}$ . In particular  $\ell \leq n \leq 2\ell - 1$ .

*Proof.* (2)  $\Rightarrow$  (1). The proof is immediate from [ST09, Proposition 2.1]. (1)  $\Rightarrow$  (2). We argue by contradiction: suppose  $X \in L_2(\mathcal{A})$  has  $[X] = \{1, \dots, p\}$ , for some  $p > 3$ . Let  $I(\mathcal{A}_X) \subset \mathbb{k}[y_1, \dots, y_p]$  be the Orlik-Terao ideal of  $\mathcal{A}_X$ . Since  $p \geq 4$ , we have four circuits on  $\{1, 2, 3, 4\}$  and four elements of  $I(\mathcal{A}_X)$ , which we abbreviate by  $r_{123}$ ,  $r_{124}$ ,  $r_{134}$ , and  $r_{234}$ . It is not difficult to see that any one of these is a linear combination of the other three, and there exists a linear syzygy on any three, so  $I(\mathcal{A}_X)$  is (minimally) generated by  $r_{1ij}$  for  $1 < i < j \leq p + 1$ .

By hypothesis,  $I(\mathcal{A})$  is generated by relations  $r_C$ , where  $|C| = 3$ . Since such circuits intersect on at most one element, monomials  $y_1 y_i$  for  $1 < i \leq p$  appear only in relations indexed by circuits of  $\mathcal{A}_X$ . It follows that  $r_{1ij}$  must be part of any minimal generating set of  $I(\mathcal{A})$  as well. Since the quadrics  $r_{123}$ ,  $r_{124}$ ,  $r_{134}$  have a linear syzygy, they cannot be part of a regular sequence, so if  $I(\mathcal{A})$  is a complete intersection, necessarily  $p \leq 3$ .

To complete the argument, note that, since rank-2 flats contain at most 3 points, the relations  $\{r_C : C \in \mathcal{C}(\mathcal{M}(\mathcal{A})), |C| = 3\}$  are linearly independent. Then, since  $I(\mathcal{A})$  is a quadratic complete intersection, the number of triple points equals  $\text{codim}(I(\mathcal{A}))$ , which is  $n - \ell$ .

(1)  $\Rightarrow$  (3): Suppose that  $R(\mathcal{A})$  is a q.c.i. Then  $\text{codim}(I(\mathcal{A})) = n - \ell$ , so  $R(\mathcal{A})$  has the Koszul graded minimal free resolution as a  $S = \mathbb{k}[y_1, \dots, y_n]$ -module

$$0 \rightarrow S(-2(n - \ell)) \rightarrow \dots \rightarrow S^{n - \ell}(-2) \rightarrow S \rightarrow R(\mathcal{A}) \rightarrow 0.$$

So the Hilbert series is

$$(17) \quad h(R(\mathcal{A}), t) = \frac{(1 - t^2)^{n - \ell}}{(1 - t)^n} = \frac{(1 + t)^{n - \ell}}{(1 - t)^\ell}.$$

By Terao’s formula (1), this is equivalent to

$$\pi(\mathcal{A}, t) = (1 + t)^{2\ell - n}(1 + 2t)^{n - \ell}.$$

Since  $\mathcal{A}$  is a central arrangement,  $1 + t$  always divides this polynomial, and therefore  $2\ell - n \geq 1$  and  $n - \ell \geq 0$ .

(3)  $\Rightarrow$  (1): Suppose we have a rank- $\ell$ , central essential arrangement with Poincaré polynomial as in (3). Then  $h(R(\mathcal{A}), t)$  is given by (17), so

$$h(R(\mathcal{A}), t) = 1 + nt + (n(n+1)/2 - (n-\ell))t^2 + \dots,$$

and  $I(\mathcal{A})$  contains  $n - \ell$  (independent) quadrics. By assumption, these generate  $I(\mathcal{A})$ , so it is a complete intersection, as in [ST09, Cor. 1.8].  $\square$

**5.2. 3-tree arrangements.** Here, we characterize the supersolvable arrangements having exponents which are at most 2. We will follow the terminology for hypergraphs of, e.g., [JMM06]. We briefly recall some definitions.

**Definition 5.2.** A hypergraph  $G = (V, E)$  is a pair of a set of vertices  $V$  and edges  $E \subseteq 2^V$ . We assume that each edge  $e \in E$  contains at least two vertices. A *walk* in  $G$  is an alternating sequence of vertices and edges with the property that consecutive vertices are contained in the intermediate edge.  $G$  is *connected* if every pair of vertices is joined by a walk. A *cycle* is a walk with at least two edges that begins and ends at the same vertex and consists of, otherwise, distinct edges and vertices. A hypergraph  $G$  with no cycles is called a *hyperforest*. A connected hyperforest is a *hypertree*. The *edge graph* of a hypergraph is the graph with vertices  $E$  and edges  $\{e, e'\}$  whenever  $e \cap e' \neq \emptyset$ .

We shall be exclusively interested in hypergraphs for which  $|e| = 3$  for each  $e \in E$ , which we will call 3-graphs, 3-trees, and 3-forests, respectively.

**Definition 5.3.** Let  $\mathcal{A}$  be an arrangement of  $n$  hyperplanes. Let  $V = [n]$ , and  $E(\mathcal{A}) = E = \{[X] : X \in L_2(\mathcal{A})\}$ . Let  $G(\mathcal{A}) := (V, E)$ , the hypergraph on codimension-2 flats of  $\mathcal{A}$ .

Note that, if  $G(\mathcal{A})$  is a 3-graph, then for any  $e, e' \in E(\mathcal{A})$ , if  $e \neq e'$ , we must have  $|e \cap e'| \leq 1$ . If  $G(\mathcal{A})$  is connected and  $|E| = 1$ , clearly  $M(\mathcal{A}) = U_{2,3}$ , the uniform matroid of 3 points in rank 2. More generally, a condition introduced by Falk [Fal02] ensures  $G(\mathcal{A})$  determines the matroid of  $\mathcal{A}$ :

**Definition 5.4.** If  $M$  is a matroid on  $[n]$ , a set  $S \subseteq [n]$  is said to be *line-closed* if  $\text{cl}(\{i, j\}) \subseteq S$  for every  $i, j \in S$ . The matroid  $M$  is *line-closed* if all line-closed sets are flats of  $M$ .

**Proposition 5.5.** If  $M(\mathcal{A})$  is line-closed and  $G(\mathcal{A})$  is a 3-graph, then a set  $S \subseteq [n]$  is a flat of  $M(\mathcal{A})$  if and only if  $|S \cap e| \neq 2$  for any  $e \in E(\mathcal{A})$ .

*Proof.* Suppose  $|S \cap e| \neq 2$  for any  $e \in E(\mathcal{A})$ . Since  $G(\mathcal{A})$  is a 3-graph, this means  $S$  is line-closed, hence a flat.  $\square$

**Theorem 5.6.** For an irreducible arrangement  $\mathcal{A}$ , the following are equivalent:

- (1)  $M(\mathcal{A})$  is line-closed and  $G(\mathcal{A})$  is a 3-tree with  $\ell - 1$  edges.
- (2)  $\mathcal{A}$  is supersolvable, with exponents  $\{1 \cdot 1, (\ell - 1) \cdot 2\}$ .
- (3)  $M(\mathcal{A})$  is an iterated parallel connection of  $\ell - 1$  uniform matroids  $U_{2,3}$ .

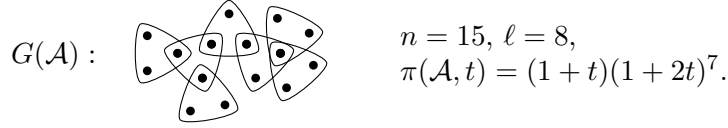


FIGURE 2. A 3-tree arrangement

*Proof.* We prove “(1)  $\Rightarrow$  (2)” by induction on  $|E|$ . The claim being trivial for  $|E| = 0$ , suppose it holds for 3-trees with  $|E| < m$ , where  $m > 0$ . Suppose  $\mathcal{A}$  has  $n$  hyperplanes and  $G(\mathcal{A})$  is a 3-tree with  $m$  edges. The edge graph of  $G(\mathcal{A})$  is a tree: let  $\{e, e'\}$  be an edge for which  $e$  has degree 1, and let  $\{i, j\} = e - e'$ . Let  $S = [n] - \{i, j\}$ . Clearly  $S$  is a maximal proper line-closed subset, so  $S = [X]$  for a coatom  $X \in L(\mathcal{A})$ .

The induced subgraph of  $G(\mathcal{A})$  on  $S$  may be identified with  $G(\mathcal{A}_X)$ . Again,  $G(\mathcal{A}_X)$  is a 3-tree, so  $\mathcal{A}_X$  is supersolvable by induction. Now  $e = \{i, j, k\}$  for some  $k$ , a circuit of  $M(\mathcal{A})$ . Then  $X \wedge (\{i\} \vee \{j\}) = \{k\}$ , so  $X$  is modular. It follows that  $\mathcal{A}$  is supersolvable, and the exponent 2 occurs with multiplicity one more than in  $\mathcal{A}_X$ .

To show “(2)  $\Rightarrow$  (3)” use induction on  $\ell$ . The base case being obvious, suppose  $\mathcal{A}_X$  is supersolvable with exponents  $\{1 \cdot 1, (\ell - 2) \cdot 2\}$  and  $X$  is modular. Let  $\{i, j\} = [n] - [X]$ , and let  $e = \text{cl}(i, j)$ . Since  $X$  is modular,  $e = \{i, j, k\}$  for some  $k \neq i, j$ . The submatroid on  $e$  is isomorphic to  $U_{2,3}$ , and the contraction  $M(\mathcal{A})/k$  is disconnected, so  $M(\mathcal{A})$  is a parallel connection of  $M(\mathcal{A}_X)$  with  $U_{2,3}$  over  $k$ , by [Oxl11, 7.1.16].

We omit the implication “(3)  $\Rightarrow$  (1)”, which is routine.  $\square$

Further discussion of the iterated parallel connection operad may be found in [DS12]. Property (3) implies that  $M(\mathcal{A})$  is graphic: the graph may be constructed by iterated parallel connection of triangles along the vertices of  $G(\mathcal{A})$ .

**5.3. A combinatorial characterization.** Now consider the hypergraph  $G(\mathcal{A})$  in the case that  $R(\mathcal{A})$  is a q.c.i. By Theorem 5.1, we see  $G(\mathcal{A})$  is a 3-graph with  $n - \ell$  edges. The quadratic complete intersection property is inherited by subarrangements:

**Lemma 5.7.** *If  $R(\mathcal{A})$  is a q.c.i., then so is  $R(\mathcal{A}_X)$  for any  $X \in L(\mathcal{A})$ .*

*Proof.* First,  $I(\mathcal{A}_X) = I_2(\mathcal{A}_X)$ : if not, suppose  $r_C$  is an irredundant generator of degree  $d \geq 3$ . By reordering the hyperplanes, we assume the circuit  $C = [d + 1]$ . Since  $R(\mathcal{A})$  is quadratic, then  $r_C = \sum_{k=1}^m P_k r_{C_k}$ , for some circuits  $C_k$  of size 3, and some homogeneous polynomials  $P_k \in \mathbb{k}[y_1, \dots, y_n]$ .

Choose a monomial order such that  $\text{Lt}(r_C) = y_1 \cdots y_d$ . Then there exist  $1 \leq i, j \leq d$ ,  $i \neq j$ , such that  $i, j \in C_k$ , for some  $1 \leq k \leq m$ . For this  $k$ ,  $|C_k \cap C| \geq 2$ . Since  $|C_k| = 3$ , this means that  $C_k \subset C$ , a contradiction.

So  $I(\mathcal{A}_X)$  is a quadratic ideal, generated by some  $\mathbb{k}$ -linear combinations of the quadratic generators of  $I(\mathcal{A})$ . A subset of a regular sequence is a regular sequence, so  $R(\mathcal{A}_X)$  is also a quadratic complete intersection.  $\square$

A reformulation of Proposition 21 of [SSV11] gives:

**Proposition 5.8.** *If  $R(\mathcal{A})$  is quadratic, then  $M(\mathcal{A})$  is line-closed.*

**Remark 5.9.** Falk showed in [Fal02] that  $M(\mathcal{A})$  is line-closed if the Orlik-Solomon algebra is quadratic, continuing our analogy. Yuzvinsky provided a counterexample to the converse: see [DY02, Ex. 4.5]. It follows from [SSV11, Prop. 21] that, if  $M(\mathcal{A})$  is line-closed, then  $Y(\mathcal{A})$  is cut out by degree-2 subideal  $I_2(\mathcal{A})$ . However, a Macaulay 2 [GS] calculation using the same example from [DY02] shows that the ideal generated by  $I_2(\mathcal{A})$  is not radical: as in the Orlik-Solomon case, then, the line-closed property does not imply quadraticity.  $\diamond$

**Lemma 5.10.** *If  $R(\mathcal{A})$  is a q.c.i., then  $G(\mathcal{A})$  is a 3-forest.*

*Proof.* Using Lemma 5.7, it is enough to show that if  $M(\mathcal{A})$  is irreducible and  $R(\mathcal{A})$  is a q.c.i., then  $G(\mathcal{A})$  is a 3-tree, since by Proposition 5.8, the connected components of  $M(\mathcal{A})$  are those of the 3-graph  $G(\mathcal{A})$ .

Let  $|G(\mathcal{A})|$  denote the simplicial complex on  $V$  with 2-simplices  $E$ . Since each 1-simplex in  $|G(\mathcal{A})|$  is contained in exactly one 2-simplex,  $|G(\mathcal{A})|$  retracts onto its edge graph, a 1-complex. For the same reason,  $|G(\mathcal{A})|$  is connected and simply connected if and only if  $G(\mathcal{A})$  is a hypertree. Computing the Euler characteristic gives

$$\begin{aligned} b_1 &= 1 - |E| + 3|E| - n, \\ &= n - 2\ell + 1, \end{aligned}$$

since  $|E(\mathcal{A})| = n - \ell$ , where  $b_i$  denotes the  $i$ th Betti number of  $|G(\mathcal{A})|$ . On the other hand, since  $M(\mathcal{A})$  is irreducible, a classical result due to Crapo implies that  $\pi'(\mathcal{A}, 1) \neq 0$ ; it follows from Theorem 5.1(3) that  $2\ell - n = 1$ , so  $b_1 = 0$ , as required.  $\square$

We conclude with a combinatorial characterization.

**Theorem 5.11.** *The following are equivalent.*

- (1)  $R(\mathcal{A})$  is a q.c.i.
- (2)  $M(\mathcal{A})$  is line-closed, and  $G(\mathcal{A})$  is a 3-forest.
- (3)  $\mathcal{A}$  is supersolvable, and its exponents are each 1 or 2.
- (4) Connected components of  $M(\mathcal{A})$  are iterated parallel connections of uniform matroids  $U_{23}$ .

*Proof.* The implication “(1)  $\Rightarrow$  (2)” is given by Lemma 5.10 and Proposition 5.8. The last three conditions are equivalent, by Theorem 5.6. If  $\mathcal{A}$  is supersolvable,  $R(\mathcal{A})$  is quadratic (Theorem 2.8), so it is enough to note that if  $\mathcal{A}$  has no exponent greater than 2, then  $\pi(\mathcal{A}, t)$  has the form of Theorem 5.1(3).  $\square$

In particular, we see that the quadratic complete intersection property depends only on the matroid of  $\mathcal{A}$ .

**Corollary 5.12.** *Suppose  $\mathcal{A}$  is an arrangement for which  $R(\mathcal{A})$  is a complete intersection. Then the implications of Corollary 4.6 are all equivalences.*

*Proof.* If  $R(\mathcal{A})$  is a q.c.i., then it is supersolvable, so it remains only to show that if  $R(\mathcal{A})$  is a complete intersection and  $\mathcal{A}$  is 2-formal, then  $R(\mathcal{A})$  is quadratic. Suppose  $I(\mathcal{A})$  is a complete intersection with minimal generating set  $\mathcal{M}$ , and let  $I_2(\mathcal{A})$  be the subideal of  $I(\mathcal{A})$  generated by degree-2 elements. By hypothesis,  $\mathcal{M}$  forms a regular

sequence. Then  $\mathcal{M}$  contains a minimal generating set  $\mathcal{N}$  for  $I_2(\mathcal{A})$ , since  $I(\mathcal{A})$  contains no elements of degree  $< 2$ .

By [ST09, Theorem 2.4], 2-formality implies  $\text{codim}(I_2(\mathcal{A})) = \text{codim } I(\mathcal{A}) = n - \ell$ . Since  $\mathcal{N}$  must also be a regular sequence, we have  $\mathcal{N} = \mathcal{M}$ , and  $I(\mathcal{A})$  is quadratic.  $\square$

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